

Partial convex hulls of coadjoint orbits and degrees of invariants

Valdemar V. Tsanov

Abstract We study properties of convex hulls of (co)adjoint orbits of compact groups, with applications to invariant theory and tensor product decompositions. The notion of partial convex hulls is introduced and applied to define two numerical invariants of a coadjoint orbit of a semisimple connected compact Lie group. It is shown that the orbits, where any one of these invariants does not exceed a given number r , form, upon intersection with a fixed Weyl chamber, a rational convex polyhedral cone in that chamber, related to the Littlewood-Richardson cone of the r -fold diagonal embedding of K . The numerical invariants are shown to provide lower bounds for degrees of invariant polynomials on irreducible unitary representations.

1 Introduction and main results

Let K be a semisimple connected compact Lie group with Lie algebra \mathfrak{k} . The negative of The Killing form of \mathfrak{k} is positive definite, denoted by (\cdot, \cdot) ; it provides an isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$. The (co)adjoint K -orbits in \mathfrak{k} are parametrized by any fixed Weyl chamber \mathfrak{t}_+ in a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{k}$. For $\lambda \in \mathfrak{t}_+$, the convex hull $\text{Conv}(K\lambda)$ is also called the orbitope of λ ; it was studied by Biliotti, Ghigi and Heinzner in [2], where all faces of this orbitope are described and shown to be exposed. In this article, we introduce the notion of partial convex hulls and derive two numerical invariants $\mathbf{r}_0(\lambda)$ and $\mathbf{r}(\lambda)$.

Valdemar V. Tsanov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, ul. "Acad. Georgi Bonchev" 8, 1113 Sofia, Bulgaria, e-mail: valdemar.tsanov@math.bas.bg.

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We show that the locus where any one of these invariants does not exceed a given $r \in \mathbb{N}$ forms a rational polyhedral convex cone in \mathfrak{t}_+ related to the Littlewood-Richardson cone of the r -fold diagonal embedding $K \hookrightarrow K^{\times r}$. The consideration of $\mathbf{r}_0(\lambda)$ and $\mathbf{r}(\lambda)$ is motivated by interpretations in terms of momentum maps and resulting applications to invariant theory and decompositions of tensor products of irreducible representations. The main results are formulated after the introduction of some notions and notation.

The r -th partial convex hull of a subset $X \subset E$ of a Euclidean space E , for a positive integer r , is defined as

$$C_r(X) := \bigcup_{x_1, \dots, x_r \in X} \text{Conv}\{x_1, \dots, x_r\}. \quad (1)$$

We focus on $X = K\lambda \subset \mathfrak{k}$, a (co)adjoint K -orbit through an arbitrary $\lambda \in \mathfrak{t}_+$. In such a case, $C_r(K\lambda)$ is preserved by K and is equal to the convex hull $\text{Conv}(K\lambda)$ for sufficiently large r , since (see Lemma 1)

$$\text{Conv}(K\lambda) = C_{\ell+1}(K\lambda), \quad \text{where } \ell := \dim \mathfrak{t}.$$

The minimal r for which $C_r(K\lambda)$ is convex will be denoted by

$$\mathbf{r}(\lambda) := \min\{r \in \mathbb{N} : \text{Conv}(K\lambda) = C_r(K\lambda)\}.$$

Since K is semisimple, $0 \in \text{Conv}(K\lambda)$ for all λ , and we define

$$\mathbf{r}_0(\lambda) := \min\{r \in \mathbb{N} : 0 \in C_r(K\lambda)\}.$$

Let $T = \exp(\mathfrak{t}) \subset K$ be the maximal torus in K corresponding to \mathfrak{t} . The character lattice $\text{Hom}(T, \mathbb{C}^\times)$ is naturally embedded in $i\mathfrak{t}^*$; we denote by $\Lambda \subset \mathfrak{t}$ its image under $\lambda \mapsto -i\lambda$ composed with the identification $\mathfrak{t}^* \cong \mathfrak{t}$ provided by (\cdot) , and refer to Λ as the weight lattice of T . Let $\Lambda^+ = \Lambda \cap \mathfrak{t}_+$ be the monoid of dominant weights. Then Λ^+ parametrizes the irreducible complex representations of K , and we denote by V_λ the irreducible representation with highest weight $i(\lambda|\cdot)$ with $\lambda \in \Lambda^+$.

Let $\mathbb{C}[V_\lambda]$ be the polynomial ring on V_λ and $\mathbb{C}[V_\lambda]_d$ denote the space of homogeneous polynomials of degree d . The ring of K -invariant polynomials $\mathbb{C}[V_\lambda]^K$ is finitely generated, by Hilbert's theorem, and the ordered sequence of the degrees $0 = d_0(\lambda) \leq \dots \leq d_p(\lambda)$ of a minimal set of homogeneous generators is the same for all such sets and thus canonically determined by $\lambda \in \Lambda^+$. Hilbert's theorem is famously nonconstructive and the degrees $d_j(\lambda)$ are generally unknown, although many special cases have subjected to extensive studies and classifications. Upper bounds for the Noether number $d_p(\lambda)$ have been derived by Popov [9] and Derksen [5]. Here we study lower bounds for the minimal positive degree $d_1(\lambda) = \min\{d \in \mathbb{Z}_{>0} : \mathbb{C}[V_\lambda]_d^K \neq 0\}$, whenever

it exists, and the variations of this degree and our bounds along variations of λ in Λ^+ . The bounds are related to the convex geometry of the orbit $K\lambda \subset \mathfrak{k}$.

The Littlewood-Richardson monoid \mathcal{LR}_r and cone \mathcal{CLR}_r of the r -fold diagonal embedding $\varphi_r : K \hookrightarrow K^{\times r}$ are defined as

$$\begin{aligned} \mathcal{LR}_r &= \mathcal{LR}_r(K) = \{(\lambda_1, \dots, \lambda_r) \in (\Lambda^+)^{\times r} : (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r})^K \neq 0\} \\ \mathcal{CLR}_r &= \mathcal{CLR}_r(K) = \text{Conv}(\mathcal{LR}_r) \\ &= \{(\lambda_1, \dots, \lambda_r) \in (\mathfrak{t}_+)^{\times r} : 0 \in K\lambda_1 + \dots + K\lambda_r\}. \end{aligned} \quad (2)$$

It is well known, see [4] for a survey of the key results and historical references, that \mathcal{LR}_r is a finitely generated monoid, \mathcal{CLR}_r is a rational polyhedral convex cone, and \mathcal{LR}_r is of finite index in the finitely generated monoid $\Lambda^{\times r} \cap \mathcal{CLR}_r$ of integral points in the cone.

The following three theorems are the main results of this article.

Theorem 1. *The inequality $\mathbf{r}_0(\lambda) \leq d_1(q\lambda)$ holds for all $\lambda \in \Lambda^+ \setminus \{0\}$ and $q \in \mathbb{Q}_{>0}$ such that $q\lambda \in \Lambda^+$.*

Let $\iota_r : \mathfrak{k} \rightarrow \mathfrak{k}^{\oplus r}, x \mapsto (x, \dots, x)$ be the differential of φ_r .

Theorem 2. *For any coadjoint orbit $K\lambda$ and $r \in \mathbb{N}$, $0 \in C_r(K\lambda)$ if and only if $0 \in K\lambda + \dots + K\lambda$ (r -fold sum). The set $\mathfrak{A}_r := \{\lambda \in \mathfrak{t}_+ : \mathbf{r}_0(\lambda) \leq r\}$ is a rational polyhedral convex cone in \mathfrak{t}_+ satisfying $\iota_r(\mathfrak{A}_r) = \mathcal{CLR}_r \cap \iota_r(\mathfrak{t}_+)$.*

Theorem 3. *The set $\mathfrak{C}_r := \{\lambda \in \mathfrak{t}_+ : \mathbf{r}(\lambda) \leq r\}$ is a rational polyhedral convex cone in \mathfrak{t}_+ . For $\lambda \in \mathfrak{t}_+$ the following are equivalent:*

- (i) $\mathbf{r}(\lambda) \leq r$, i.e., $C_r(K\lambda)$ is convex;
- (ii) the r -fold sum $K\lambda + \dots + K\lambda$ is convex;
- (iii) for every $\xi \in \mathfrak{t}_+$, $\lambda|_{\mathfrak{k}'_\xi} \in \mathcal{CLR}_r(K'_\xi)$, where K'_ξ is the derived subgroup of the centralizer of ξ , \mathfrak{k}'_ξ is its Lie algebra and $\lambda|_{\mathfrak{k}'_\xi}$ is the orthogonal projection of λ to \mathfrak{k}'_ξ .

Further details and corollaries from the above theorems will be given in the main text. In §2 we recall the interpretation of the Littlewood-Richardson cone in terms of momentum maps, we derive properties of \mathbf{r}_0 and prove Theorems 1 and 2. In §3 we describe a relationship between \mathbf{r}_0 and \mathbf{r} , prove Theorem 3 and compute the values of \mathbf{r}_0 and \mathbf{r} at fundamental weights of classical groups.

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2 The momentum map and the Littlewood-Richardson cone

We begin by recalling Heckman's interpretation of projections of coadjoint orbits as momentum maps, [6], and the resulting interpretation of the Littlewood-Richardson monoid and cone in the framework of the Geometric Invariant Theory (GIT) of Hilbert-Mumford and Kirwan, [7]. The survey [4] contains more details and historical references for the background results outlined here. The first basic fact is that every coadjoint orbit $K\lambda \subset \mathfrak{k}^* \cong \mathfrak{k}$ of a compact connected Lie group K admits a canonical invariant Kähler structure, called the Kostant-Kirillov-Souriau structure, so that the inclusion $K\lambda \subset \mathfrak{k}^*$ is a K -equivariant momentum map. The underlying complex manifolds of the coadjoint orbits are known as the flag varieties of K . Two orbits are equivariantly isomorphic as complex manifolds if and only if their points of intersection with a fixed Weyl chamber \mathfrak{t}_+ belong to the relative interior of the same face of \mathfrak{t}_+ . By the Borel-Weil theorem, for integral $\lambda \in \Lambda^+$, the Kähler structure corresponds to a K -linearized holomorphic line bundle \mathcal{L}_λ with space of global sections isomorphic to the irreducible representation V_λ^* . Moreover $K\lambda$ is equivariantly isomorphic to the projective orbit of a highest weight vector $K[v_\lambda] \subset \mathbb{P}(V_\lambda)$ with the Kähler structure induced by the Fubini-Study form of a suitably normalized K -invariant Hermitean form on V_λ . The homogeneous coordinate ring of the projective variety $K[v_\lambda]$, i.e., the ring of sections of \mathcal{L}_λ , is

$$R_\lambda := \bigoplus_{q \in \mathbb{Z}_{\geq 0}} H^0(K\lambda, \mathcal{L}_\lambda^q) \cong \bigoplus_{q \in \mathbb{Z}_{\geq 0}} V_{q\lambda}^* .$$

Heckman's theorem, [6], states that, if $K \subset \tilde{K}$ is an embedding of compact connected Lie groups, $\iota : \mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}}$ is the corresponding inclusion of Lie algebras, and $\iota^* : \tilde{\mathfrak{k}}^* \rightarrow \mathfrak{k}^*$ is the dual map, then the restriction of ι^* to the coadjoint \tilde{K} -orbit $Z_{\tilde{\lambda}} = \tilde{K}\tilde{\lambda}$ through any fixed $\tilde{\lambda} \in \tilde{\mathfrak{k}}^*$,

$$\mu = \mu_K^{\tilde{\lambda}} := (\iota_r^*)|_{Z_{\tilde{\lambda}}} : Z_{\tilde{\lambda}} \rightarrow \mathfrak{k}^* \cong \mathfrak{k}$$

is a K -equivariant momentum map with respect to the canonical \tilde{K} -equivariant Kostant-Kirillov-Souriau structure on $Z_{\tilde{\lambda}}$. This puts us in the setting of Kirwan's Geometric Invariant Theory, [7], and we have the following two fundamental results. First, the intersection of the momentum image with a Weyl chamber, $\mu(Z_{\tilde{\lambda}}) \cap \mathfrak{t}_+$, is a convex polytope, called the momentum polytope. Second, for integral $\tilde{\lambda} \in \tilde{\Lambda}^+$, the symplectic reduction $\mu^{-1}(0)/K$ can be identified with the GIT-quotient, which is an algebraic variety isomorphic to the projective spectrum of the invariant ring $(R_{\tilde{\lambda}})^K$. This invariant ring can in turn be interpreted by applying the Borel-Weil theorem to \tilde{K} :

$$\mu^{-1}(0)/K \cong \text{Proj}(R_{\tilde{\lambda}}^K), \quad R_{\tilde{\lambda}}^K \cong \bigoplus_{q \in \mathbb{Z}_{\geq 0}} (V_{q\tilde{\lambda}}^*)^K.$$

Since $V_{\tilde{\lambda}}^K \neq 0$ if and only if $(V_{\tilde{\lambda}}^*)^K \neq 0$, we have the equivalences

$$0 \in \mu(Z_{\tilde{\lambda}}) \iff R_{\tilde{\lambda}}^K \neq \mathbb{C} \iff \exists q \in \mathbb{N} : (V_{q\tilde{\lambda}})^K \neq 0. \quad (3)$$

The set

$$\mathcal{CLR}(K \subset \tilde{K}) = \{\tilde{\lambda} \in \tilde{\mathfrak{t}}_+ : 0 \in \mu(Z_{\tilde{\lambda}})\}$$

is a rational polyhedral convex cone, known as the generalized Littlewood-Richardson cone, described by inequalities derived from the Hilbert-Mumford functions of suitable one-parameter subgroups of K . These inequalities have a long history culminating with Ressayre's characterization of a minimal list of inequalities, [10]. We shall not need the exact form of the inequalities, and satisfy ourselves with the aforementioned properties of the cone. The set of integral points $\tilde{A} \cap \mathcal{CLR}(K \subset \tilde{K})$ is a finitely generated monoid containing as a finitely generated submonoid of finite index the so-called generalized Littlewood-Richardson monoid

$$\mathcal{LR}(K \subset \tilde{K}) = \{\tilde{\lambda} \in \tilde{A}^+ : (V_{\tilde{\lambda}})^K \neq 0\}.$$

Thus there exists $q \in \mathbb{N}$ such that $q(\tilde{A} \cap \mathcal{CLR}(K \subset \tilde{K})) \subset \mathcal{LR}(K \subset \tilde{K})$. Furthermore, we have $\mathcal{CLR}(K \subset \tilde{K}) = \text{Conv}(\mathcal{LR}(K \subset \tilde{K}))$. We refer the reader to [4] for more details.

Here we consider what is in fact a prototypical case for the above construction: the diagonal subgroup $K \subset \tilde{K} := K^{\times r}$ of a r -fold Cartesian product for some fixed $r \in \mathbb{N}$. On the level of Lie algebras we have the inclusion $\iota_r : \mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}} := \mathfrak{k}^{\oplus r}$, $x \mapsto (x, \dots, x)$. Every Weyl chamber \mathfrak{t}_+ of \mathfrak{k} is contained in unique Weyl chamber of $\tilde{\mathfrak{k}}$, namely $\tilde{\mathfrak{t}}_+ = (\mathfrak{t}_+)^{\times r}$. We also have $\tilde{A} = A^{\times r}$. The irreducible representations of \tilde{K} are tensor products of irreducible representations of K , i.e., for $\tilde{\lambda} = (\lambda_1, \dots, \lambda_r) \in \tilde{A}^+$, $V_{\tilde{\lambda}} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r}$. Therefore, the space $V_{\tilde{\lambda}}^K$ of K -invariant vectors in an irreducible \tilde{K} -module is, in this setting, the space of invariant tensors in a tensor product of r irreducible K -modules.

For $\tilde{\lambda} = (\lambda_1, \dots, \lambda_r) \in \tilde{\mathfrak{t}}_+$ denote by $Z_{\tilde{\lambda}} = \tilde{K}\tilde{\lambda} = K\lambda_1 \times \dots \times K\lambda_r \subset \tilde{\mathfrak{k}}$ the (co)adjoint \tilde{K} -orbit through $\tilde{\lambda}$ equipped with its canonical Kostant-Kirillov-Souriau Kähler structure.

To apply Heckman's theorem, we note that the dual map ι_r^* is given by

$$\iota_r^* : \tilde{\mathfrak{k}} \cong \tilde{\mathfrak{k}}^* \cong (\mathfrak{k}^*)^{\oplus r} \rightarrow \mathfrak{k}^* \cong \mathfrak{k}, \quad (x_1, \dots, x_r) \mapsto x_1 + \dots + x_r.$$

and so

$$\mu = \mu_{\tilde{K}}^{\tilde{\lambda}} = (\iota_r^*)|_{Z_{\tilde{\lambda}}} : Z_{\tilde{\lambda}} \rightarrow \mathfrak{k}$$

is a momentum map for the diagonal K -action on $Z_{\tilde{\lambda}}$. The momentum image is

$$\mu(Z_{\tilde{\lambda}}) = K\lambda_1 + K\lambda_2 + \dots + K\lambda_r.$$

Thus we can apply the GIT framework explained above. First, by Kirwan's theorem, $\mu(Z_{\tilde{\lambda}}) \cap \mathfrak{t}_+$ is a convex polytope. Second, we can interpret the Littlewood-Richardson cone \mathcal{CLR}_r defined in the Introduction as

$$\mathcal{CLR}_r = \mathcal{CLR}(K \subset K^{\times r}) = \{(\lambda_1, \dots, \lambda_r) \in (\mathfrak{t}_+)^{\times r} : 0 \in \mu(Z_{(\lambda_1, \dots, \lambda_r)})\}.$$

The set of integral points of this cone is

$$\tilde{A} \cap \mathcal{CLR}_r = \{(\lambda_1, \dots, \lambda_r) \in (A^+)^{\times r} : \exists q \in \mathbb{N}, (V_{q\lambda_1} \otimes \dots \otimes V_{q\lambda_r})^K \neq 0\}.$$

and is a finitely generated monoid containing \mathcal{LR}_r as a submonoid of finite index. In particular, for $\tilde{\lambda} = (\lambda_1, \dots, \lambda_r) \in \tilde{A}^+$, the equivalence (3) takes the form

$$0 \in \mu(Z_{\tilde{\lambda}}) \iff \exists q \in \mathbb{N} : (V_{q\lambda_1} \otimes V_{q\lambda_2} \otimes \dots \otimes V_{q\lambda_r})^K \neq 0.$$

With this preparation, we turn to our applications.

Theorem 4. *For any (co)adjoint orbit $K\lambda \in \mathfrak{k}$ and $r \in \mathbb{N}$, $0 \in C_r(K\lambda)$ if and only if $0 \in K\lambda + \dots + K\lambda \subset \mathfrak{k}$ (r -fold sum). Furthermore, the set $\mathfrak{A}_r := \{\lambda \in \mathfrak{t}_+ : \mathbf{r}_0(\lambda) \leq r\}$ is a rational polyhedral convex cone in \mathfrak{t}_+ satisfying $\iota_r(\mathfrak{A}_r) = \mathcal{CLR}_r \cap \iota_r(\mathfrak{t}_+)$.*

Proof. We consider the \tilde{K} -homogeneous complex manifold $Z = (K/K_\lambda)^{\times r}$, which is \tilde{K} -equivariantly isomorphic to $(K\lambda)^{\times r}$ as a complex manifold. This manifold can be endowed with different Kähler structures and K -momentum maps arising from the different \tilde{K} -equivariant embeddings of Z into $\tilde{\mathfrak{k}}$; these embeddings are obtained as $Z \cong \tilde{K}\tilde{\lambda}' \subset \tilde{\mathfrak{k}}$ for $\tilde{\lambda}'$ belonging to the relative interior of the same face of $\tilde{\mathfrak{t}}_+$ as $(\lambda, \dots, \lambda)$. It is convenient to take into consideration cases, where instead of an embedding of Z we have a map whose image is isomorphic to $(K/K_\lambda)^{\times s}$ for some $s \leq r$. For $\lambda \in \mathfrak{t}_+$ and $\underline{q} = (q_1, \dots, q_r) \in (\mathbb{R}_{\geq 0})^{\times r} \setminus \{0\}$, we denote $\underline{q} \cdot \lambda := (q_1\lambda, \dots, q_r\lambda) \in \tilde{\mathfrak{t}}_+$ and $Z_{\underline{q} \cdot \lambda} := \tilde{K}\tilde{\lambda} = q_1 K\lambda \times \dots \times q_r K\lambda \subset \tilde{\mathfrak{k}}$. Then, by Heckman's theorem,

$$\mu^{\underline{q} \cdot \lambda} := \frac{1}{\sum q_j} (\iota_r^*)|_{Z_{\underline{q} \cdot \lambda}} : Z_{\underline{q} \cdot \lambda} \rightarrow \mathfrak{k}^* \cong \mathfrak{k}$$

is a momentum map for the diagonal K -action on $Z_{\underline{q} \cdot \lambda}$ and the Kostant-Kirillov-Souriau Kähler structure multiplied by a factor of $\frac{1}{\sum q_j}$. We introduce this factor, because we want the image of $\mu^{\underline{q} \cdot \lambda}$ to be contained in the partial convex hull $C_r(K\lambda)$ defined in (1). We observe that $C_r(K\lambda)$ consists of the images of all such momentum maps, i.e.,

$$C_r(K\lambda) = \bigcup_{\underline{q} \in (\mathbb{R}_{\geq 0})^{\times r} \setminus \{0\}} \mu^{\underline{q}, \lambda}(Z_{\underline{q}, \lambda}).$$

Hence, the condition $0 \in C_r(K\lambda)$ can now be reformulated in the following equivalent forms:

- (a) $r \geq \mathbf{r}_0(\lambda)$;
- (b) $\exists \underline{q} = (q_1, \dots, q_r) \in (\mathbb{R}_{\geq 0})^{\times r} \setminus \{0\}$ such that $0 \in \mu^{\underline{q}, \lambda}(Z_{\underline{q}, \lambda})$;
- (c) $\exists \underline{q} = (q_1, \dots, q_r) \in (\mathbb{R}_{\geq 0})^{\times r} \setminus \{0\}$ such that $(q_1\lambda, \dots, q_r\lambda) \in \mathcal{CLR}_r$.

Let us notice that the convex cone $\mathcal{CLR}_r \subset (\mathfrak{t}_+)^{\times r}$ is stable under permutations of the components, as can be seen directly from the definition (2). This implies that condition (c) is equivalent to $(\lambda, \dots, \lambda) \in \mathcal{CLR}_r$. Indeed, if such \underline{q} exists, then all permutations $(q_{\sigma(1)}\lambda, \dots, q_{\sigma(r)}\lambda)$ belong to \mathcal{CLR}_r and so does their sum, but this sum is positively proportional to $(\lambda, \dots, \lambda)$, so $(\lambda, \dots, \lambda) \in \mathcal{CLR}_r$. We obtain

$$r \geq \mathbf{r}_0(\lambda) \iff (\lambda, \dots, \lambda) \in \mathcal{CLR}_r,$$

which completes the proof of the theorem.

Next, we derive a corollary concerning the degrees of K -invariant elements of the polynomial ring $\mathbb{C}[V_\lambda]$ over the irreducible representation space V_λ of K . Recall that $\mathbb{C}[V_\lambda]_d$ denotes the space of homogeneous polynomials of degree d ; this space can be interpreted as the d -th symmetric tensor power of the dual K -module V_λ^* . Furthermore, we have $(S^d V_\lambda)^* \cong S^d(V_\lambda^*)$. The dual module V_λ^* is irreducible and we denote by λ^* its highest weight, so that $V_\lambda^* \cong V_{\lambda^*}$.

Corollary 5 *Let $\lambda \in \Lambda^+ \setminus \{0\}$ and $b_1(\lambda) = \min\{b \in \mathbb{N} : \exists q \in \mathbb{N} : \mathbb{C}[V_{q\lambda}]_b^K \neq 0\}$. Then the inequality $b_1(\lambda) \geq \mathbf{r}_0(\lambda)$ holds.*

Proof. According to the remarks leading to the corollary, we have $\mathbb{C}[V_\lambda]_r = S^r V_\lambda^*$ and $\mathbb{C}[V_\lambda]_r^* = S^r V_\lambda \cong \mathbb{C}[V_{\lambda^*}]_r$. Since any given K -module has K -invariant vectors if and only if its dual has K -invariant vectors, we have $b_1(\lambda) = b_1(\lambda^*)$ and we may consider $S^r V_\lambda$ instead of $\mathbb{C}[V_\lambda]_r$. Now, the existence of $q \in \mathbb{N}$ such that $(S^r V_{q\lambda})^K \neq 0$ clearly implies the existence of $q \in \mathbb{N}$ such that $(V_{q\lambda}^{\otimes r})^K \neq 0$, which in turn is equivalent to $(\lambda, \dots, \lambda) \in \mathcal{CLR}_r$. But we have shown in Theorem 4 that the latter is equivalent to $r \geq \mathbf{r}_0(\lambda)$, and this completes the proof.

Remark 2.1 *Let $\lambda \in \mathfrak{t}_+$, $r \in \mathbb{N}$ and*

$$\mathfrak{Q}_{\lambda, r} := \{(q_1\lambda, \dots, q_r\lambda) \in (\mathfrak{t}_+)^{\times r} : q_j \in \mathbb{R}_{\geq 0} \forall j\},$$

which is a simplicial cone in $(\mathfrak{t}_+)^{\times r}$. Then $\mathfrak{Q}_{\lambda, r} \not\subseteq \mathcal{CLR}_r$ and

$$r \geq \mathbf{r}_0(\lambda) \iff \mathfrak{Q}_{\lambda, r} \cap \mathcal{CLR}_r \neq \{0\} \iff \mathfrak{Q}_{\lambda, r} \cap \partial(\mathcal{CLR}_r) \neq \{0\}.$$

Consequently, if $\lambda \in \Lambda^{++} := \Lambda \cap \text{Relint}(\mathfrak{t}_+)$ is strictly dominant and $r = \mathbf{r}_0(\lambda)$, then $\mathfrak{Q}_{\lambda, r}$ intersects a regular face of \mathcal{CLR}_r . It would be interesting

to understand which regular faces of \mathcal{CLR}_r can be attained in this way, i.e., intersect $\mathfrak{Q}_{\lambda,r}$ for some $\lambda \in \Lambda^{++}$.

3 Coadjoint orbitopes and partial convex hulls

Here we establish some basic properties of the partial convex hulls $C_r(K\lambda)$ of coadjoint K -orbits, derive combinatorial interpretations for $\mathbf{r}_0(\lambda)$ and $\mathbf{r}(\lambda)$, and compute the values for all fundamental weights of classical groups. We also recall the description of the faces of the orbitope $\text{Conv}(K\lambda)$ by Biliotti, Ghigi and Heinzner, [2], as it is used in an essential way in the our approach to $\mathbf{r}(\lambda)$.

Let $\Delta \subset \Lambda$ be the root system of K with respect to T , split as $\Delta = \Delta^+ \sqcup \Delta^-$ by the chosen Weyl chamber \mathfrak{t}_+ , and let $\Pi \subset \Delta^+$ be the set of simple roots. Let $W = N_K(T)/T$ be the Weyl group acting on \mathfrak{t} as the group generated by the (simple) root reflections. Recall that $K\lambda \cap \mathfrak{t} = W\lambda$ for every $\lambda \in \mathfrak{t}$. By a classical theorem of Kostant, if $\iota_T^* : \mathfrak{k} \rightarrow \mathfrak{t}$ is the orthogonal projection, then $\iota_T^*(K\lambda) = \text{Conv}(W\lambda) = \text{Conv}(K\lambda) \cap \mathfrak{t}$. Consequently,

$$\text{Conv}(K\lambda) = K\text{Conv}(W\lambda) .$$

It turn out that all faces of $\text{Conv}(K\lambda)$ arise from faces of the polytope $\text{Conv}(W\lambda)$. We continue with necessary notation.

The complexified Lie algebra $\mathfrak{k}^c = \mathfrak{k} \oplus i\mathfrak{k}$ is a complex semisimple Lie algebra with root space decomposition:

$$\mathfrak{k}^c = \mathfrak{t}^c \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C}e_\alpha \right) .$$

The root vectors can be chosen so that

$$\mathfrak{k} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta^+} (\mathbb{R}(e_\alpha - e_{-\alpha}) \oplus \mathbb{R}i(e_\alpha + e_{-\alpha})) \right) .$$

For $\xi \in \mathfrak{t}$, the centralizer subalgebra $\mathfrak{k}_\xi = \mathfrak{z}_\mathfrak{k}(\xi)$ is given by

$$\mathfrak{k}_\xi = \mathfrak{t} \oplus \mathfrak{k} \cap \left(\bigoplus_{\alpha \in \Delta : (\alpha|\xi)=0} \mathbb{C}e_\alpha \right) , \quad .$$

The corresponding centralizer subgroup of K is connected, and we have $K_\xi = Z_K(\xi) = \exp(\mathfrak{k}_\xi)$. For $\xi \in \mathfrak{t}_+$, \mathfrak{k}_ξ is uniquely determined by the set of simple roots vanishing on ξ , $\Pi^\xi = \{\alpha \in \Pi : (\alpha|\xi) = 0\}$. Conversely, any subset $\hat{\Pi} \subset \Pi$ gives rise to a subalgebra $\mathfrak{k}_{\hat{\Pi}} \subset \mathfrak{k}$ with center $\mathfrak{z}_{(\hat{\Pi})} := \mathfrak{z}(\mathfrak{k}_{(\hat{\Pi})}) = \bigcap_{\alpha \in \hat{\Pi}} \ker \alpha$.

Theorem 6. (*Biliotti-Ghigi-Heinzner [2]*) *Let $\lambda \in \mathfrak{t}_+$. Then $\text{Conv}(K\lambda) = K\text{Conv}(W\lambda)$, the faces of $\text{Conv}(K\lambda)$ are exposed and are exactly the subsets*

of the form

$$\text{Conv}(gK_{(\hat{H})}\lambda) = gK_{(\hat{H})}\text{Conv}(W_{(\hat{H})}\lambda)$$

for $\hat{H} \subset H$ and $g \in K$.

With this preparation, we now turn our attention to the partial convex hulls and begin with the following elementary observation. Recall that $\ell = \dim \mathfrak{t}$ denotes the rank of \mathfrak{k} .

Lemma 1. *For any $\lambda \in \mathfrak{t}_+$ we have $\mathbf{r}(\lambda) \leq \ell + 1$, i.e., $C_{\ell+1}(K\lambda) = \text{Conv}(K\lambda)$.*

Proof. Since $\text{Conv}(K\lambda) = K\text{Conv}(W\lambda)$, it suffices to show that $C_{\ell+1}(W\lambda) = \text{Conv}(W\lambda)$.

Lemma 2. *Let $S = \{x_1, \dots, x_n\} \in E$ be a finite set of points in a real vector space E , such that the affine hull of S is the entire E , and let $m = \dim E$. Then $C_{m+1}(S) = \text{Conv}(S)$.*

Proof. We may assume, without loss of generality, that S is the set of extreme points of its convex hull $P = \text{Conv}(S)$. We have to show that every point $x \in P$ belongs to some simplex of the form $\text{Conv}\{x_{j_1}, \dots, x_{j_{m+1}}\}$. We shall proceed by induction on n . For $n = 1$ the statement is trivially true. Assume that it holds for sets of cardinality $n - 1$ or less. Let F be any facet of P . Then the induction hypothesis implies $F = C_m(F \cap S)$. Hence, if $x_j \notin F$, then $\text{Conv}(F \cup \{x_j\}) = C_{m+1}(\{x_j\} \cup (F \cap S))$. Now let us fix one of the points, say x_1 . Observe that P is equal to the union of segments connecting x_1 to the boundary of P . Furthermore, it suffices to consider on the the facet which do not contain the fixed point. Let F_1, \dots, F_k be the facets of P which do not contain x_1 . We have

$$\begin{aligned} P &= \text{Conv}(\{x_1\} \cup F_1 \cup \dots \cup F_k) = \bigcup_{j=1}^k \text{Conv}(\{x_1\} \cup F_j) \\ &= \bigcup_{j=1}^k C_{m+1}(\{x_1\} \cup (F_j \cap S)) \subset C_{m+1}(S). \end{aligned}$$

This completes the proof of Lemma 2.

We apply this result to obtain

$$\text{Conv}(K\lambda) = K\text{Conv}(W\lambda) = KC_{\ell+1}(W\lambda) \subset C_{\ell+1}(K\lambda),$$

and this completes the proof of Lemma 1.

It is easy to see that $r(\lambda) = 1$ if and only if $\mathbf{r}_0(\lambda) = 1$, if and only if $\lambda = 0$.

Remark 3.1 Let $w_0 \in W$ denote the longest Weyl group element, characterised by $w_0(t_+) = -t_+$. For $\lambda \in \mathfrak{t}_+$, the dominant weight $\lambda^* = -w_0\lambda$ is called the dual weight to λ , since for $\lambda \in \Lambda^+$ we have $V_{\lambda^*} \cong V_{\lambda}^*$. In general, $K(\lambda^*) = K(-\lambda)$ holds and implies

$$\lambda = \lambda^* \iff \mathbf{r}_0(\lambda) = 2.$$

Recall that for the simple groups of type $B_\ell, C_\ell, D_{2m}, E_7, E_8, F_4, G_2$ one has $w_0 = -1$. Hence, in these cases, $\mathbf{r}_0(\lambda) = 2$ for all $\lambda \in \mathfrak{t}_+$. Thus the behaviour of \mathbf{r}_0 is nontrivial only if K has simple factors of type A_m, D_{2m+1}, E_6 .

In the examples below we compute the value of \mathbf{r}_0 for all fundamental weights of classical simple groups, which are not self dual, i.e., the spin-representations of $Spin_{4\ell}$ with odd ℓ and the fundamental representations of $SU_{\ell+1}$.

Example 1. Let $K = SU_n$ and let $\varpi_1, \dots, \varpi_{n-1}$ be the fundamental weights of K in their standard order ($\ell = n - 1$). We have

$$\mathbf{r}_0(\varpi_1) = n \quad , \quad \mathbf{r}_0(\varpi_2) = \begin{cases} \frac{n}{2}, & \text{for } n \text{ even} \\ \frac{n+3}{2}, & \text{for } n \text{ odd} \end{cases}.$$

If $n = jk$ with $j \leq n/2$, we have $\mathbf{r}_0(\varpi_j) = k$.

If $n = jq + p$ with $p \in \{0, \dots, j-1\}$, then one can show that $\mathbf{r}_0(\varpi_{j, SU_n}) = q + \mathbf{r}_0(\varpi_{p, SU_j})$, which by induction yields

$$\mathbf{r}_0(\varpi_j) = \sum_{k=1}^m q_k,$$

where m is the number of steps in the Euclidean algorithm for n, j (at the last step we get rest 0) and q_k is the integral part of the k -th quotient. For instance, for $j = 3$, we obtain

$$\mathbf{r}_0(\varpi_3) = \left\lceil \frac{n}{3} \right\rceil = \begin{cases} q & , \text{ if } n = 3q \\ q + 3 & , \text{ if } n = 3q + 1 \text{ or } 3q + 2 \end{cases}.$$

Example 2. Let $K = Spin_{2\ell}$ with ℓ odd, $\ell \geq 3$. Then

$$\mathbf{r}_0(\varpi_{\ell-1}) = \mathbf{r}_0(\varpi_\ell) = 4.$$

To see this, observe that the weights of the spin-representation are of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with an even number of minus signs. For $\ell = 3$ a minimal set $M \subset W\lambda$ whose convex hull contains 0 is

$$M = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \right\}.$$

For larger ℓ we just extend the above weights in a trivial way, two times with positive signs and two times with negative, in order to get zero as a sum.

For any subset $\hat{H} \subset \Pi$ we denote by $k'_{(\hat{H})}$ the semisimple part (i.e., derived subalgebra) of $\mathfrak{k}_{(\hat{H})}$. Then $\mathfrak{t}_{(\hat{H})} = \mathfrak{t} \cap k'_{(\hat{H})} = \text{span}\{\alpha^\vee : \alpha \in \hat{H}\}$ is a maximal abelian subalgebra of $k'_{(\hat{H})}$. For $\lambda \in \mathfrak{t}_+$ we denote by $\lambda|_{\hat{H}}$ is the projection of λ to $\mathfrak{t}_{(\hat{H})}$, which defines a dominant weight of $\mathfrak{k}'_{(\hat{H})}$.

Theorem 7. *For $\lambda \in \mathfrak{t}_+$, the following holds:*

$$\mathbf{r}(\lambda) = \max\{\mathbf{r}_0(\lambda|_{\hat{H}}) : \hat{H} \subset \Pi\}.$$

Proof. Recall that the center of $\mathfrak{k}_{(\hat{H})}$ is given by $\mathfrak{z}_{(\hat{H})} = \cap_{\alpha \in \hat{H}} \ker \alpha$. Since $\lambda \in \mathfrak{t} \subset \mathfrak{k}_{(\hat{H})} \subset \mathfrak{k}$ the orbit $K_{(\hat{H})}\lambda$ can be seen as a coadjoint $K_{(\hat{H})}$ -orbit. In particular, the semisimple part $K'_{(\hat{H})}$ acts transitively on it, and the intersection of $\text{Conv}(K_{(\hat{H})}\lambda)$ with $\mathfrak{z}_{(\hat{H})}$ is a single point which we denote by $\nu_{\lambda, \hat{H}}$, so that

$$\{\nu_{\lambda, \hat{H}}\} = \mathfrak{z}_{(\hat{H})} \cap \text{Conv}(W_{(\hat{H})}\lambda) = \mathfrak{z}_{(\hat{H})} \cap \text{Conv}(K_{(\hat{H})}\lambda).$$

Furthermore, $\nu_{\lambda, \hat{H}} \in \mathfrak{t}_+$, i.e. it is dominant, and the set $E(\lambda) := \{\nu_{\lambda, \hat{H}} : \hat{H} \subset \Pi\}$ is exactly the extreme points of the polytope $\mathfrak{t}_+ \cap \text{Conv}(W\lambda)$.

For any subset $\hat{H} \subset \Pi$, we have

$$\mathbf{r}_0(\lambda|_{\hat{H}}) = \min\{r \in \mathbb{N} : \nu_{\lambda, \hat{H}} \in C_r(K_{(\hat{H})}\lambda)\} = \min\{r \in \mathbb{N} : \nu_{\lambda, \hat{H}} \in C_r(K\lambda)\},$$

where the second equality holds, since, due to Theorem 6, $\text{Conv}(K_{(\hat{H})}\lambda)$ is an exposed face of $\text{Conv}(K\lambda)$ containing $\nu_{\lambda, \hat{H}}$. Hence

$$\max\{\mathbf{r}_0(\lambda|_{\hat{H}}) : \hat{H} \subset \Pi\} = \min\{r \in \mathbb{N} : E(\lambda) \subset C_r(K\lambda)\} \leq \mathbf{r}(\lambda).$$

On the other hand, recall the momentum map

$$\mu^{\iota_r(\lambda)} : (K\lambda)^{\times r} \rightarrow \mathfrak{k}, (x_1, \dots, x_r) \mapsto \frac{1}{r}(x_1 + \dots + x_r),$$

whose image $\mu^{\iota_r(\lambda)}((K\lambda)^{\times r}) = \frac{1}{r}(K\lambda + \dots + K\lambda)$ is contained in $C_r(K\lambda)$. Since the intersection of the momentum image with the Weyl chamber \mathfrak{t}_+ is convex, to prove the theorem it is sufficient to show that $E(\lambda) \subset \mu^{\iota_r(\lambda)}((K\lambda)^{\times r})$ for $r = \max\{\mathbf{r}_0(\lambda|_{\hat{H}}) : \hat{H} \subset \Pi\}$. From Theorem 4 we know that $r \geq \mathbf{r}_0(\lambda)$ is equivalent to $0 \in \frac{1}{r}(K\lambda + \dots + K\lambda)$. Applying this to each $K_{(\hat{H})}\lambda|_{\hat{H}}$ we get $r \geq \mathbf{r}_0(\lambda|_{\hat{H}})$ if and only if $\nu_{\lambda, \hat{H}} \in \frac{1}{r}(K_{(\hat{H})}\lambda|_{\hat{H}} + \dots + K_{(\hat{H})}\lambda|_{\hat{H}})$, which yields the desired result.

Now we are ready to prove Theorem 3 stated in the Introduction.

Proof (of Theorem 3). We shall first show the equivalence of the three conditions. Note that condition (iii) of the theorem can be rewritten with the

notation introduced in this section as: for every subset $\hat{I} \subset \Pi$, $\lambda_{|\hat{I}} \in \mathcal{CLR}_r(K'_{(\hat{I})})$. From Theorem 4 we know that $\lambda_{|\hat{I}} \in \mathcal{CLR}_r(K'_{(\hat{I})})$ is equivalent to $r \geq \mathbf{r}_0(\lambda_{|\hat{I}})$. Thus condition (iii) is equivalent to $r \geq \mathbf{r}_0(\lambda_{|\hat{I}})$ for all $\hat{I} \subset \Pi$, which is in turn equivalent to $r \geq \mathbf{r}(\lambda)$ due to Theorem 7. Therefore conditions (i) and (iii) are equivalent.

The fact that (ii) implies (i) follows immediately from the observation that the rescaled r -fold sum $\frac{1}{r}(K\lambda + \dots + K\lambda)$ is contained in $C_r(K\lambda)$ and contains $K\lambda$, hence if the sum is convex then so is the partial convex hull and we must have $r \geq \mathbf{r}(\lambda)$.

We suppose now that (i) (and hence (iii)) holds and shall verify (ii), or more precisely, that the normalized r -fold sum $A_r = \frac{1}{r}(K\lambda + \dots + K\lambda)$ is convex. By Theorem 4, $0 \in C_r(K\lambda)$ is equivalent to $0 \in A_r$. Similarly, from the proof of Theorem 7 and with the notation introduced therein, we deduce that $\nu_{\lambda, \hat{I}} \in C_r(K\lambda)$ is equivalent to $\nu_{\lambda, \hat{I}} \in C_r(K_{(\hat{I})}\lambda)$, which is in turn equivalent to $\nu_{\lambda, \hat{I}} \in \frac{1}{r}(K_{(\hat{I})}\lambda + \dots + K_{(\hat{I})}\lambda) \subset A_r$. Thus (i) implies $\nu_{\lambda, \hat{I}} \in A_r$ for all $\hat{I} \subset \Pi$. The convex hull of these points is equal to $\mathfrak{t}_+ \cap \text{Conv}(K\lambda)$, and A_r must contain this convex hull, because A_r is equal to the momentum image $\mu^{\iota_r(\lambda)}(K\lambda^{\times r})$ and hence its intersection with the Weyl chamber is a convex polytope. Thus $A_r = \text{Conv}(K\lambda)$ and property (ii) holds.

It remains to show that \mathfrak{C}_r is a rational convex polyhedral cone. It is not difficult to see that, for any $\hat{I} \subset \Pi$, the set

$$\mathfrak{B}_{r, \hat{I}} = \{(\lambda_1, \dots, \lambda_r) \in (\mathfrak{t}_+)^{\times r} : ((\lambda_1)_{|\hat{I}}, \dots, (\lambda_r)_{|\hat{I}}) \in \mathcal{CLR}_r(K'_{(\hat{I})})\}$$

is a rational polyhedral convex cone described analogously to the Littlewood-Richardson cone. Now the equivalence between (i) and (iii) yields

$$\mathfrak{C}_r = \iota_r(\mathfrak{t}_+) \cap \left(\bigcap_{\hat{I} \subset \Pi} \mathfrak{B}_{r, \hat{I}} \right)$$

and we can deduce the claimed properties of \mathfrak{C}_r .

As a special case, we have the following.

Corollary 8 *For $\lambda \in \mathfrak{t}_+ \setminus \{0\}$ the following are equivalent:*

- (i) $\mathbf{r}(\lambda) = 2$;
- (ii) $K\lambda + K\lambda$ is convex;
- (iii) for any subset $\hat{I} \subset \Pi$, $\mathbf{r}_0(\lambda_{|\hat{I}}) = 2$, i.e., $K'_{(\hat{I})}\lambda_{|\hat{I}} \ni -\lambda_{|\hat{I}}$;
- (iv) For any connected component Π_1 of (the Dynkin diagram of) Π and any $\alpha, \beta \in \Pi_1$ such that $\|\alpha\| = \|\beta\|$, the equality $(\lambda|\alpha) = (\lambda|\beta)$ holds.

Example 3. Let K be a simply connected classical group and $\varpi_1, \dots, \varpi_\ell$ be the fundamental weights of K with the standard ordering of [3]. The values of \mathbf{r} at the fundamental weights can be computed using Theorem 7 and the

values of \mathbf{r}_0 at fundamental weights computed in Remark 3.1 and Examples 1, 2. Here we use the fact that the simple factors of centralizer subgroups in classical groups are again classical groups. We obtain the following.

For $K = SU_{\ell+1}$ and $1 \leq j \leq \frac{\ell+1}{2}$ we have $\mathbf{r}(\varpi_j) = \mathbf{r}(\varpi_{\ell+1-j}) = (\ell+1) - (j-1)$.

For $K = Spin_{2\ell+1}$, or $Sp_{2\ell}$, we have $\mathbf{r}(\varpi_j) = \mathbf{r}(\varpi_{\ell-j}) = \ell - (j-1)$, for $1 \leq j \leq \frac{\ell}{2}$, and $\mathbf{r}(\varpi_\ell) = 2$.

For $K = Spin_{2\ell}$ we have $\mathbf{r}(\varpi_j) = \mathbf{r}(\varpi_{\ell-j}) = \ell - (j-1)$, for $1 \leq j \leq \frac{\ell}{2}$, and $\mathbf{r}(\varpi_\ell) = \ell$.

We observe that $\mathbf{r}(\varpi_j)$ is equal to $1 + \hat{\ell}_j$, where $\hat{\ell}_j$ is the length of the longest chain of type A in Π emanating from α_j . The number $\hat{\ell}_j$ is also equal to the dimension of the largest projective space \mathbb{P} equivariantly embedded in $K\varpi_j$ and mapped to a linear subspace in $\mathbb{P}(V_{\omega_j})$ when $K\varpi_j$ is mapped (isomorphically) to the orbit $K[v_{\varpi_j}]$ of a highest weight line (see [8]).

Remark 3.2 *Lists of inequalities describing the cones \mathfrak{A}_r and \mathfrak{C}_r (see Theorems 2 and 3) can be deduced from the known lists of inequalities describing \mathcal{CLR}_r . A minimal list of inequalities for \mathcal{CLR}_r was obtained by Belkale and Kumar in [1]. The resulting inequalities for \mathfrak{A}_r have the form*

$$(\lambda|w_1\varpi_j + \dots + w_r\varpi_j) \geq 0,$$

where ϖ_j is a fundamental weight of K and $w_1, \dots, w_r \in W$ satisfy the so-called Belkale-Kumar condition (cf. [1], [4]).

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